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COMMENT

Comments on a recent note on the Schrödinger equation with a δ'-interaction

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Abstract. It is argued that a recent letter by Bao-Heng Zhao on the one-dimensional Schrödinger equation with a δ'-interaction is seriously flawed and hence arrives at wrong conclusions.

In the recent letter [1], Bao-Heng Zhao studied the one-dimensional Schrödinger equation with a δ'-interaction, i.e.

$$-\psi''(x) + c\delta'(x)\psi(x) = E\psi(x) \qquad x \in \mathbb{R} \tag{1}$$

with c a coupling constant, and concluded that (1) can be replaced by the free Schrödinger equation

$$-\psi''(x) = E\psi(x) \tag{2}$$

supplemented with the 'boundary conditions'

$$\psi(0^+) = \psi(0^-) = \psi(0) = 0 \tag{3a}$$

$$\psi'(0^+) - \psi'(0^-) = -c\psi'(0)\,. \tag{3b}$$

(Actually the author in [1], in addition to (3b), also mentions the boundary condition $\psi'(0^+) - \psi'(0) = -c\psi'(0)$ in his equation (2) but this appears to be a typographical error.)

In order to arrive at (3a), (3b), the author in [1] makes use of the distributional relation

$$\delta'(x)\psi(x) = \delta'(x)\psi(0) - \delta(x)\psi'(0)\,. \tag{4}$$

Given the conditions (3a), (3b), the author in [1] then goes on and claims that the following boundary conditions:

$$\begin{aligned} &\psi(0^+) - \psi(0^-) = c\psi'(0) \qquad c \in \mathbb{R} \\ &\psi'(0^+) = \psi'(0^-) \end{aligned} \tag{5}$$

employed in references [2, 3], are 'irrelevant' in connection with (1).

Here we would like to point out the following facts which prove that the reasoning in [1] is seriously flawed and hence wrong conclusions have been reached:

(i) Condition (3*a*) by itself is already the boundary condition for a unique self-adjoint extension of the minimal operator $H_{0,\min} = -\mathrm{d}^2/\mathrm{d}x^2$ defined on the domain $C_0^\infty(\mathbb{R}\backslash\{0\})$ in the usual Hilbert space $L^2(\mathbb{R})$. In fact, (3*a*) represents precisely the boundary condition of the self-adjoint Dirichlet extension H_0^D of $H_{0,\min}$, where

$$H_0^D = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$
$$\mathcal{D}(H_0^D) = \{\psi \in H^{2,1}(\mathbb{R}) \cap H^{2,2}(\mathbb{R}\backslash\{0\}) | \psi(0) = 0\}. \tag{6}$$

(Note that $\psi \in H^{2,1}(\mathbb{R})$ implies $\psi(0^+) = \psi(0^-) = \psi(0)$ and hence the boundary condition in (6) is idential to (3*a*). Here $H^{m,n}(\mathbb{R}), m, n \in \mathbb{N}$ denote the usual Sobolev spaces.) The operator H_0^D in (6) is precisely the direct sum of the two Dirichlet Laplacians on $(0, \infty)$ and $(-\infty, 0)$, respectively.

Due to this elementary fact any additional boundary condition, such as (3*b*), together with (3*a*) necessarily represents a *non-self-adjoint* operator which is entirely unacceptable as a Hamiltonian in a quantum mechanical context. On the contrary, the boundary condition (5) used in [2, 3], defines a family of *self-adjoint* extensions of $H_{0,\min}$.

(ii) Equation (3*b*) is ill-defined as it stands since the symbol $\psi'(0)$ is not explained in [1]. More importantly, however, equation (3*b*) is based on the distributional relation (4) which clearly requires at least the differentiability of ψ at $x = 0$. Hence if taken seriously, equation (3*b*) either leads to the trivial case $c = 0$, i.e. to $\psi'(0^+) = \psi'(0^-) = \psi'(0)$, or to $\psi'(0^+) = \psi'(0^-) = \psi'(0) = 0$. As is stressed in (i) above, both conditions, when combined with (3*a*), yield non-self-adjoint operators in $L^2(\mathbb{R})$. We also note the trivial fact that $\psi'(0)$ does not exist in general (and hence the use of (4) is not permitted) as the standard theory of self-adjoint extensions of $H_{0,\min}$ (whose deficiency indices are (2,2)) readily reveals. This is clearly reflected in the Dirichlet extension H_0^D in (6).

Moreover, in the special case $c = 0$ the boundary conditions (3*a*), (3*b*) (in contrast to the case $c = 0$ in (5)) do *not* reduce to the free kinetic energy operator $H_0 = -\mathrm{d}^2/\mathrm{d}x^2$ on the domain $H^{2,2}(\mathbb{R})$.

(iii) Finally, we would like to point out that, as has been stressed in [3, appendix G], the δ'-interaction in [1] should not be taken too literally. In fact, when considering this interaction in momentum space, the δ'-interaction, in contrast to the δ-interaction in one-dimension, but similary to the two- and three-dimensional δ-interactions, requires a certain coupling constant renormalization procedure. For the three-dimensional δ-interaction this is of course well known and goes back to a celebrated paper by Berezin and Faddeev [4]. Analogous considerations in connection with (1) then lead to the self-adjoint boundary conditions (5) and hence justify their use in this context [2, 3, 5, 6].

References

[1] Bao-Heng Zhao 1992 *J. Phys. A: Math. Gen.* **25** L617

[2] Gesztesy F and Holden H 1987 *J. Phys. A: Math. Gen.* **20** 5157

[3] Albeverio S, Gesztesy H, Høegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Berlin: Springer)

[4] Berezin F A and Faddeev L D 1961 *Sov. Math. Dokl.* **2** 372

[5] Grossmann A, Høegh-Krohn R and Mebkhout M 1980 *J. Math. Phys.* **21** 2376

[6] Šeba P 1988 *Rep. Math. Phys.* **24** 111